

A CONSTRUCTION OF COMMUTING SYSTEMS OF INTEGRABLE SYMPLECTIC BIRATIONAL MAPS. LIE-POISSON CASE

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ABSTRACT. We give a construction of completely integrable $(2n)$ -dimensional Hamiltonian systems with symplectic brackets of the Lie-Poisson type (linear in coordinates) and with quadratic Hamilton functions. Applying to any such system the so called Kahan-Hirota-Kimura discretization scheme, we arrive at a birational $(2n)$ -dimensional map. We show that this map is symplectic with respect to a symplectic structure that is a perturbation of the original symplectic structure on \mathbb{R}^{2n} , and possesses n independent integrals of motion, which are perturbations of the original Hamilton functions and are in involution with respect to the invariant symplectic structure. Thus, this map is completely integrable in the Liouville-Arnold sense. Moreover, under a suitable normalization of the original n -tuples of commuting vector fields, their Kahan-Hirota-Kimura discretizations also commute and share the invariant symplectic structure and the n integrals of motion. This paper extends our previous ones, arXiv:1606.08238 [nlin.SI] and arXiv:1607.07085 [nlin.SI], where similar results were obtained for Hamiltonian systems with a constant (canonical) symplectic structure and cubic Hamilton functions.

1. INTRODUCTION

In the recent papers [9,10], we introduced a large family of integrable symplectic maps, appearing as the so called Kahan-Hirota-Kimura discretization of a big family of completely integrable Hamiltonian systems in arbitrary even dimension $2n$, with the canonical symplectic structure and cubic Hamilton functions. We mentioned there an open problem of generalizing these findings for the case of Hamiltonian systems with a Poisson tensor linear in local coordinates (that is, a Lie-Poisson tensor) and quadratic Hamilton functions. Here, such a generalization is achieved.

We consider a certain family of Lie-Poisson tensors $J(x)$ of full rank (thus, defining symplectic structures) on \mathbb{R}^{2n} . For such a tensor, there exist constant $2n \times 2n$ matrices A satisfying

$$A^T J(x) = J(x) A, \quad \forall x \in \mathbb{R}^{2n}. \quad (1)$$

Clearly, any power of A satisfies the same equation. Generically, along with A , one has an n -dimensional vector space of matrices satisfying (1) which consists of polynomials of A of degree $n - 1$. To each non-degenerate matrix A with property (1), there corresponds a vector space of quadratic polynomials $H_0(x)$ on \mathbb{R}^{2n} , satisfying a system of second order

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linear PDEs encoded in the matrix equation

$$A(\nabla^2 H) = (\nabla^2 H)A^T, \quad (2)$$

where $\nabla^2 H$ is the Hesse matrix of the function H . Such a polynomial $H_0(x)$ can be included to an n -tuple of quadratic polynomials $H_i(x)$ satisfying the same matrix differential equations (2), and characterized by

$$\nabla H_i(x) = A \nabla H_{i-1}(x), \quad i = 1, \dots, n-1.$$

For a generic A , the functions $H_i(x)$, $i = 0, \dots, n-1$, are functionally independent and are in involution with respect to the Lie-Poisson structure on \mathbb{R}^{2n} defined by the tensor $J(x)$. Thus, the flows of the Hamiltonian vector fields $f_i(x) = J(x) \nabla H_i(x)$ commute, and comprise a completely integrable Hamiltonian system.

When applied to a completely integrable Hamiltonian system $\dot{x} = f_0(x)$ of this family, the Kahan-Hirota-Kimura discretization method produces the map Φ_{f_0} with the following striking properties.

- The map Φ_{f_0} is Poisson with respect to a (symplectic) Poisson structure on \mathbb{R}^{2n} which is a perturbation of the original one (defined by $J(x)$), and possesses n functionally independent integrals in involution. In other words, the map Φ_{f_0} is completely integrable.
- In general, the maps Φ_{f_i} do not commute among themselves. However, one can find systems of commuting maps which include Φ_{f_0} . We say that a linear combination of the vector fields,

$$\sum_{i=0}^{n-1} \alpha_i f_i(x) = J(x) \left(\sum_{i=0}^{n-1} \alpha_i A^i \right) \nabla H_0(x) = J(x) B \nabla H_0(x) = B^T f_0(x),$$

is *associated* to the vector field $f_0(x)$, if the matrix $B = \sum_{i=0}^{n-1} \alpha_i A^i$ satisfies $B^2 = I$. Equivalently, the polynomial $B(\lambda) = \sum_{i=0}^{n-1} \alpha_i \lambda^i$ sends each of the n distinct eigenvalues of A to ± 1 . This defines an equivalence relation on the set of vector fields $J(x) \nabla H(x)$ with $H(x)$ satisfying (2). It turns out that Kahan-Hirota-Kimura discretizations of associated vector fields commute and share the invariant symplectic structure and n functionally independent integrals.

- The common integrals $\tilde{H}(x, \varepsilon)$ of the Kahan-Hirota-Kimura discretizations of the associated vector fields are rational perturbations of the original polynomial Hamilton functions

$$H(x) = \sum_{i=0}^{n-1} \alpha_i H_i(x),$$

and satisfy the same second order differential equation (2) as $H(x)$ do.

We give a quick review of the Kahan-Hirota-Kimura discretization method in Sect. 2. The specialization of this method for Lie-Poisson systems is considered in Sect. 3. Then, in Sect. 4, we discuss details of the general construction of completely integrable Hamiltonian systems generated by a Lie-Poisson tensor $J(x)$ and a matrix A related as in (1). Algebraic properties of the corresponding vector fields are collected in Sect. 5. Associated vector fields are introduced in Sect. 6. We prove the main results in Sect. 7 (commutativity), Sect. 8 (integrals of motion), Sect. 9 (invariant symplectic structure) and Sect. 10 (differential equations for the integrals of the maps).

2. GENERAL PROPERTIES OF THE KAHAN-HIROTA-KIMURA DISCRETIZATION

Here we recall the main facts about the Kahan-Hirota-Kimura discretization.

This method was introduced in the geometric integration literature by Kahan in the unpublished notes [4] as a method applicable to any system of ordinary differential equations on \mathbb{R}^N with a quadratic vector field:

$$\dot{x} = f(x) = Q(x) + Bx + c,$$

where each component of $Q : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a quadratic form, while $B \in \text{Mat}_{N \times N}(\mathbb{R})$ and $c \in \mathbb{R}^N$. Kahan's discretization (with stepsize 2ε) reads as

$$\frac{\tilde{x} - x}{2\varepsilon} = 2f\left(\frac{x + \tilde{x}}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(\tilde{x}) = Q(x, \tilde{x}) + \frac{1}{2}B(x + \tilde{x}) + c, \quad (3)$$

where

$$Q(x, \tilde{x}) = \frac{1}{2}(Q(x + \tilde{x}) - Q(x) - Q(\tilde{x}))$$

is the symmetric bilinear form corresponding to the quadratic form Q . Equation (3) is *linear* with respect to \tilde{x} and therefore defines a *rational* map $\tilde{x} = \Phi_f(x, \varepsilon)$. Clearly, this map approximates the time 2ε shift along the solutions of the original differential system. Since equation (3) remains invariant under the interchange $x \leftrightarrow \tilde{x}$ with the simultaneous sign inversion $\varepsilon \mapsto -\varepsilon$, one has the *reversibility* property

$$\Phi_f^{-1}(x, \varepsilon) = \Phi_f(x, -\varepsilon). \quad (4)$$

In particular, the map f is *birational*. The explicit form of the map Φ_f defined by (3) is

$$\tilde{x} = \Phi_f(x, \varepsilon) = x + 2\varepsilon (I - \varepsilon f'(x))^{-1} f(x), \quad (5)$$

where $f'(x)$ denotes the Jacobi matrix of $f(x)$. Moreover, if the vector field $f(x)$ is homogeneous (of degree 2), then (5) can be equivalently rewritten as

$$\tilde{x} = \Phi_f(x, \varepsilon) = (I - \varepsilon f'(x))^{-1} x. \quad (6)$$

Due to (4), in the latter case we also have:

$$x = \Phi_f(\tilde{x}, -\varepsilon) = (I + \varepsilon f'(\tilde{x}))^{-1} \tilde{x} \Leftrightarrow \tilde{x} = (I + \varepsilon f'(\tilde{x})) x. \quad (7)$$

One has the following expression for the Jacobi matrix of the map Φ_f :

$$d\Phi_f(x) = \frac{\partial \tilde{x}}{\partial x} = (I - \varepsilon f'(x))^{-1} (I + \varepsilon f'(\tilde{x})). \quad (8)$$

Kahan applied this discretization scheme to the famous Lotka-Volterra system and showed that in this case it possesses a very remarkable non-spiralling property. This property was explained by Sanz-Serna [11] by demonstrating that in this case the numerical method preserves an invariant Poisson structure of the original system.

The next intriguing appearance of this discretization was in two papers by Hirota and Kimura who (being apparently unaware of the work by Kahan) applied it to two famous *integrable* system of classical mechanics, the Euler top and the Lagrange top [3, 5]. Surprisingly, the discretization scheme produced in both cases *integrable* maps.

In [6–8] the authors undertook an extensive study of the properties of the Kahan's method when applied to integrable systems (we proposed to use in the integrable context the term “Hirota-Kimura method”). It was demonstrated that, in an amazing number of

cases, the method preserves integrability in the sense that the map $\Phi_f(x, \varepsilon)$ possesses as many independent integrals of motion as the original system $\dot{x} = f(x)$.

Further remarkable geometric properties of the Kahan's method were discovered by Celledoni, McLachlan, Owren and Quispel in [1, 2]. They demonstrated that for an arbitrary Hamiltonian vector field $f(x) = J\nabla H(x)$ with a constant Poisson tensor J and a cubic Hamilton function $H(x)$, the map $\Phi_f(x, \varepsilon)$ possesses a rational integral of motion as well as an invariant measure with a rational density. These properties are unrelated to integrability.

Finally, in papers [9, 10] which are direct precursors of the present one, there were discovered commuting families of completely integrable Hirota-Kimura maps, along with their full sets of integrals and with an invariant symplectic structure.

3. GENERAL PROPERTIES OF THE KAHAN-HIROTA-KIMURA DISCRETIZATION APPLIED TO LIE-POISSON TYPE SYSTEMS

In this paper, we consider the following class of vector fields on \mathbb{R}^N :

$$f(x) = J(x)\nabla H(x), \quad (9)$$

where $J(x)$ is a $N \times N$ matrix whose entries are linear forms in x , and $H(x)$ is a quadratic form in x . We have:

$$f'(x) = J(x)\nabla^2 H + J'\nabla H(x). \quad (10)$$

Here $\nabla^2 H$ is the (constant) Hesse matrix of $H(x)$, J' is a (constant) tensor such that, for any $y \in \mathbb{R}^N$, $J'y$ is the $N \times N$ matrix with the entries

$$(J'y)_{ik} = \sum_{j=1}^N \frac{\partial J_{ij}(x)}{\partial x_k} y_j.$$

From this we derive the following identity which we mention for the later reference:

$$(J'y)z = J(z)y, \quad y, z \in \mathbb{R}^N. \quad (11)$$

Proposition 1. *For any vector field of the form (9), the Kahan map $\tilde{x} = \Phi_f(x)$ can be implicitly written as*

$$\tilde{x} - x = 2\varepsilon \left(I - \varepsilon J(x)\nabla^2 H \right)^{-1} J \left(\frac{x + \tilde{x}}{2} \right) \nabla H(x), \quad (12)$$

or, alternatively, as

$$\tilde{x} = \left(I - \varepsilon J(x)\nabla^2 H \right)^{-1} \left(I + \varepsilon J(\tilde{x})\nabla^2 H \right) x. \quad (13)$$

Proof. From (10) we derive:

$$I - \varepsilon f'(x) = \left(I - \varepsilon J(x)\nabla^2 H \right) \left(I - \varepsilon (I - \varepsilon J(x)\nabla^2 H)^{-1} J' \nabla H(x) \right).$$

Therefore, equation (5) can be rewritten as

$$\tilde{x} - x = 2\varepsilon \left(I - \varepsilon (I - \varepsilon J(x)\nabla^2 H)^{-1} J' \nabla H(x) \right)^{-1} \left(I - \varepsilon J(x)\nabla^2 H \right)^{-1} f(x).$$

Multiplying this equation with the inverse of the first factor on the right-hand side, and taking into account equations (11) and (9), we find:

$$\tilde{x} - x - \varepsilon \left(I - \varepsilon J(x)\nabla^2 H \right)^{-1} J(\tilde{x} - x) \nabla H(x) = 2\varepsilon \left(I - \varepsilon J(x)\nabla^2 H \right)^{-1} J(x) \nabla H(x).$$

There follows, due to linearity of $J(x)$:

$$\tilde{x} - x = \varepsilon \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} J(\tilde{x} + x) \nabla H(x),$$

which coincides with (12).

From the latter equation we further derive, taking into account that $\nabla H(x) = (\nabla^2 H)x$:

$$\begin{aligned} \tilde{x} &= \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} \left(I - \varepsilon J(x) \nabla^2 H + \varepsilon J(\tilde{x} + x) \nabla^2 H \right) x \\ &= \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right) x, \end{aligned}$$

which finishes the proof. \square

4. A FAMILY OF INTEGRABLE LIE-POISSON SYSTEMS

Starting from this point, we always set $N = 2n$. Thus, we consider the phase space \mathbb{R}^{2n} with coordinates $x = (x_1, \dots, x_{2n})^T$. We write

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, \quad u = (x_1, \dots, x_n)^T, \quad v = (x_{n+1}, \dots, x_{2n})^T. \quad (14)$$

Equip the phase space with the Lie-Poisson tensor

$$J(x) = \begin{pmatrix} 0 & X(u) \\ -X(u) & 0 \end{pmatrix}, \quad (15)$$

where

$$X(u) = X^T(u) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & x_1 & \cdots & x_{n-1} \end{pmatrix} \quad (16)$$

is an $n \times n$ cyclic Hankel (therefore symmetric) matrix. The rank of $J(x)$ at a generic point $x \in \mathbb{R}^{2n}$ is equal to $2n$. To check that $J(x)$ is a Poisson tensor, one has to verify the Jacobi identity

$$\pi_{ijk} = \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\} = 0 \quad (17)$$

for all possible triples of indices $\{i, j, k\}$ from $\{1, 2, \dots, 2n\}$. Due to the block-diagonal structure of the matrix $J(x)$, one only has to check this for the cases $i, j \in \{1, \dots, n\}$, $k \in \{n+1, \dots, 2n\}$ and $i, j \in \{n+1, \dots, 2n\}$, $k \in \{1, \dots, n\}$. Moreover, in these cases (17) simplifies to

$$\pi_{ijk} = \{x_i, \{x_j, x_k\}\} - \{x_j, \{x_i, x_k\}\} = 0. \quad (18)$$

Both terms on the right-hand side vanish if $i, j \in \{1, \dots, n\}$, $k \in \{n+1, \dots, 2n\}$. In the remaining case $i, j \in \{n+1, \dots, 2n\}$, $k \in \{1, \dots, n\}$, we compute:

$$\{x_j, x_k\} = -x_{j+k-1 \pmod{n}},$$

and further

$$\{x_i, \{x_j, x_k\}\} = -\{x_i, x_{j+k-1 \pmod{n}}\} = x_{i+j+k-2 \pmod{n}}.$$

Since this expression is symmetric with respect to $i \leftrightarrow j$, we see that (18) is satisfied.

For any function $H_0(x)$ on \mathbb{R}^{2n} , the corresponding Hamiltonian system is governed by the equations of motion

$$\dot{x} = J(x)\nabla H_0(x).$$

Proposition 2. *Consider a constant non-degenerate $2n \times 2n$ matrix A , and suppose that for the functions $H_1(x), \dots, H_{n-1}(x)$ the following relations are satisfied:*

$$\nabla H_i(x) = A\nabla H_{i-1}(x), \quad i = 1, \dots, n-1. \quad (19)$$

If the matrix A satisfies

$$A^T J(x) = J(x) A \quad (20)$$

for any $x \in \mathbb{R}^{2n}$, then the functions $H_i(x)$ are pairwise in involution.

Proof. Let $0 \leq i < j \leq n-1$. Then $\nabla H_j = A^{j-i}\nabla H_i$. We have:

$$\{H_i, H_j\} = (\nabla H_i)^T J(x) \nabla H_j = (\nabla H_i)^T J(x) A^{j-i} \nabla H_i.$$

Since for any $\ell \in \mathbb{N}$ the matrix A^ℓ satisfies the same condition as (20), that is,

$$(A^\ell)^T J(x) = J(x) A^\ell,$$

that is, $J(x)A^\ell$ is skew-symmetric, we conclude that $\{H_i, H_j\} = 0$. \square

If the minimal annihilating polynomial of the matrix A has degree n , then the matrices I, A, \dots, A^{n-1} are linearly independent, and then equation (19) ensures that H_0, \dots, H_{n-1} are generically functionally independent. Effectively, we are considering a family of functions $H(x)$ such that

$$\nabla H(x) = (\beta_0 I + \beta_1 A + \dots + \beta_{n-1} A^{n-1}) \nabla H_0(x). \quad (21)$$

We now discuss applicability of this construction. For a given function H_0 , differential equations (19) for H_1 are solvable if and only if H_0 satisfies the following condition:

$$A(\nabla^2 H_0) = (\nabla^2 H_0) A^T, \quad (22)$$

where $\nabla^2 H_0$ is the Hesse matrix of the function H_0 . If this condition is satisfied, then for solutions of (19) we find:

$$A(\nabla^2 H_1) = A^2(\nabla^2 H_0) = A(\nabla^2 H_0) A^T = (\nabla^2 H_1) A^T.$$

Thus, H_1 satisfies the same condition (22). By induction, the same is true for all H_i , thus for all functions H satisfying (21).

Proposition 3. *For the matrix $J(x)$ given in (15), (16), the set of matrices A satisfying (20) is given by*

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix} = \mathcal{A} \oplus \mathcal{A}, \quad (23)$$

where \mathcal{A} is a circulant $n \times n$ matrix.

Proof. We write

$$J(x) = x_1 J_1 + x_2 J_2 + \dots + x_n J_n, \quad \text{where} \quad J_k = \begin{pmatrix} 0 & Q_k \\ -Q_k & 0 \end{pmatrix}, \quad (24)$$

and Q_k are $n \times n$ cyclic Hankel matrices with the entries

$$(Q_k)_{i,j} = \begin{cases} 1, & i+j = k+1 \pmod{n}, \\ 0, & \text{else.} \end{cases} \quad (25)$$

Condition (20) is equivalent to

$$A^T J_k = J_k A, \quad k = 1, \dots, n. \quad (26)$$

If A is written in the block form as

$$A = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix},$$

with $n \times n$ blocks \mathcal{A}_i , then condition (20) reads:

$$\mathcal{A}_1^T Q_k = Q_k \mathcal{A}_4, \quad (27)$$

$$\mathcal{A}_2^T Q_k + Q_k \mathcal{A}_2 = 0, \quad \mathcal{A}_3^T Q_k + Q_k \mathcal{A}_3 = 0 \quad (28)$$

for all $k = 1, \dots, n$. One easily shows that:

- Conditions (28) force the matrices $\mathcal{A}_2, \mathcal{A}_3$ to vanish. Indeed, we find:

$$(\mathcal{A}_2)_{i,j} = -(\mathcal{A}_2)_{k+1-j, k+1-i} \quad \text{for all } k = 1, \dots, n.$$

For $k = i + j - 1$ we arrive at $(\mathcal{A}_2)_{i,j} = 0$.

- Conditions (27) force that the matrix $\mathcal{A}_1 = \mathcal{A}_4$ is circulant. Indeed, we find:

$$(\mathcal{A}_1)_{i,j} = (\mathcal{A}_4)_{k+1-j, k+1-i} \quad \text{for all } k = 1, \dots, n.$$

For $k = i + j - 1$ this gives $(\mathcal{A}_1)_{i,j} = (\mathcal{A}_4)_{i,j}$. Moreover, for $k = i + j + \ell - 1$, we arrive at $(\mathcal{A}_1)_{i,j} = (\mathcal{A}_4)_{i+\ell, j+\ell} = (\mathcal{A}_1)_{i+\ell, j+\ell}$. Thus, shifting both indices i, j by the same amount does not change the value of $(\mathcal{A}_1)_{i,j}$.

We finally arrive at (23). □

For any such matrix \mathcal{A} , the set of all linear combinations $\beta_0 I + \beta_1 \mathcal{A} + \dots + \beta_{n-1} \mathcal{A}^{n-1}$ is a vector subspace of the space of circulant matrices. Thus, without restricting the generality, we can set from the beginning:

$$A = \begin{pmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{P} \end{pmatrix} = \mathcal{P} \oplus \mathcal{P}, \quad \mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (29)$$

\mathcal{P} is the $n \times n$ cyclic shift matrix, satisfying $\mathcal{P}^T = \mathcal{P}^{-1} = \mathcal{P}^{n-1}$.

With this matrix A , the quadratic solutions of the matrix differential equation (22) are easily characterized.

Proposition 4. *A quadratic form*

$$H_0(x) = \frac{1}{2} \sum_{i,j=1}^{2n} h_{ij} x_i x_j$$

with a constant symmetric matrix $\nabla^2 H_0 = (h_{ij})_{i,j=1}^{2n}$ satisfies matrix differential equation (22) with the matrix A from (29) if and only if

$$\nabla^2 H_0 = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 \\ \mathcal{H}_2 & \mathcal{H}_3 \end{pmatrix}, \quad (30)$$

where $n \times n$ blocks $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are cyclic Hankel (therefore symmetric) matrices. The dimension of the space of solutions of (22) is $3n$.

Proof. Clearly, matrix equation (22) is equivalent to $\mathcal{P}\mathcal{H}_k = \mathcal{H}_k\mathcal{P}^T$ for $k = 1, 2, 3$. This is equivalent to $(\mathcal{H}_k)_{i,j} = (\mathcal{H}_k)_{i-1,j+1}$, so that $(\mathcal{H}_k)_{i,j}$ only depends on $i + j$, that is, \mathcal{H}_k is cyclic Hankel. \square

The two classes of $n \times n$ matrices: *circulant* ones, characterized by $\mathcal{P}\mathcal{A} = \mathcal{A}\mathcal{P}$, and *cyclic Hankel* ones, characterized by $\mathcal{P}\mathcal{A} = \mathcal{A}\mathcal{P}^{-1}$, will play a fundamental role in our work. For further reference, we give an important property of the latter class.

Lemma 5. *If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are three cyclic Hankel matrices, then*

$$\mathcal{A}\mathcal{B}\mathcal{C} = \mathcal{C}\mathcal{B}\mathcal{A}, \quad (31)$$

and this product is a cyclic Hankel matrix.

Proof. Since all three matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are symmetric, equation (31) is equivalent to the statement that the matrix $\mathcal{A}\mathcal{B}\mathcal{C}$ is symmetric. But this is a consequence of the fact that this matrix is cyclic Hankel, which follows from

$$\mathcal{P}(\mathcal{A}\mathcal{B}\mathcal{C})\mathcal{P} = \mathcal{P}\mathcal{A}\mathcal{P} \cdot \mathcal{P}^{-1}\mathcal{B}\mathcal{P}^{-1} \cdot \mathcal{P}\mathcal{C}\mathcal{P} = \mathcal{A}\mathcal{B}\mathcal{C}. \quad \square$$

Note that (31) is easily generalized for a product of any odd number of cyclic Hankel matrices.

5. GENERAL ALGEBRAIC PROPERTIES OF THE VECTOR FIELDS $f_i(x) = J(x)\nabla H_i(x)$

Assume that $H_0(x)$ is a homogeneous quadratic polynomial satisfying (22). Set

$$f_0(x) = J(x)\nabla H_0(x). \quad (32)$$

Lemma 6. *Vector field $f_0(x)$ satisfies the following identity:*

$$A^T f'_0(x) = f'_0(x) A^T. \quad (33)$$

Proof. We prove that both terms on the right-hand side of (10) satisfy (33) separately. For the first term we use (22):

$$A^T J(x) \nabla^2 H_0(x) = J(x) A \nabla^2 H_0(x) = J(x) \nabla^2 H_0(x) A^T.$$

For the second term, we prove a more general statement, namely

$$A^T (J'y) = (J'y) A^T$$

for an arbitrary vector

$$y = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^{2n}, \quad \xi, \eta \in \mathbb{R}^n.$$

Here, according to (24), $J'y$ is the $2n \times 2n$ matrix whose first n columns are given by $J_k y$, $k = 1, \dots, n$, while the last n columns vanish, so that

$$J'y = (J_1 y \ \cdots \ J_n y \ 0 \ \cdots \ 0) = \begin{pmatrix} Q_1 \eta & \cdots & Q_n \eta & 0 & \cdots & 0 \\ -Q_1 \xi & \cdots & -Q_n \xi & 0 & \cdots & 0 \end{pmatrix}. \quad (34)$$

Thus, according to (29), relation $A^T(J'y) = (J'y)A^T$ is equivalent to

$$\mathcal{P}^T (Q_1 \eta \ \cdots \ Q_n \eta) = (Q_1 \eta \ \cdots \ Q_n \eta) \mathcal{P}^T, \quad (35)$$

$$\mathcal{P}^T (Q_1 \xi \ \cdots \ Q_n \xi) = (Q_1 \xi \ \cdots \ Q_n \xi) \mathcal{P}^T. \quad (36)$$

One easily sees that

$$(Q_1 \xi \ \cdots \ Q_n \xi) = \xi_1 I + \xi_2 \mathcal{P}^{-1} + \xi_3 \mathcal{P}^{-2} + \dots + \xi_n \mathcal{P}^{-n+1} \quad (37)$$

is a circulant matrix, therefore it commutes with $\mathcal{P}^T = \mathcal{P}^{-1}$. This proves (35), (36). \square

Now let $H_1(x)$ be a function satisfying $\nabla H_1(x) = A \nabla H_0(x)$, and set

$$f_1(x) = J(x) \nabla H_1(x) = J(x) A \nabla H_0(x). \quad (38)$$

Due to (20), we have:

$$f_1(x) = A^T f_0(x). \quad (39)$$

By differentiation of (39) we have:

$$f'_1(x) = A^T f'_0(x). \quad (40)$$

Corollary 7. *The following identities hold true:*

$$f'_0(x) f_1(x) = f'_1(x) f_0(x), \quad (41)$$

$$f'_0(x) f'_1(x) = f'_1(x) f'_0(x). \quad (42)$$

Proof. We compute with the help of (39), (33):

$$f'_0(x) f_1(x) = f'_0(x) A^T f_0(x) = A^T f'_0(x) f_0(x) = f'_1(x) f_0(x),$$

and similarly, with the help of (40), (33):

$$f'_0(x) f'_1(x) = f'_0(x) A^T f'_0(x) = A^T f'_0(x) f'_0(x) = f'_1(x) f'_0(x).$$

Note that (41) expresses the commutativity of the vector fields $f_0(x)$ and $f_1(x)$. \square

Lemma 8. *We have:*

$$(J(x) \nabla^2 H)^2 = q_0(x) I + q_1(x) A + \dots + q_{n-1}(x) A^{n-1}, \quad (43)$$

where

$$\begin{aligned} q_0(x) &= \frac{1}{2n} \operatorname{tr} \left((J(x) \nabla^2 H)^2 \right), \\ q_\ell(x) &= \frac{1}{2n} \operatorname{tr} \left((A^T)^\ell (J(x) \nabla^2 H)^2 \right) \\ &= \frac{1}{2n} \operatorname{tr} (J(x) \nabla^2 H_\ell \cdot J(x) \nabla^2 H), \quad \ell = 1, \dots, n-1. \end{aligned}$$

Proof. One easily computes from (15), (30):

$$J(x)\nabla^2 H = \begin{pmatrix} X\mathcal{H}_2 & X\mathcal{H}_3 \\ -X\mathcal{H}_1 & -X\mathcal{H}_2 \end{pmatrix},$$

where for the sake of brevity we write X for $X(u)$, and further:

$$(J(x)\nabla^2 H)^2 = \begin{pmatrix} X\mathcal{H}_2X\mathcal{H}_2 - X\mathcal{H}_3X\mathcal{H}_1 & X\mathcal{H}_2X\mathcal{H}_3 - X\mathcal{H}_3X\mathcal{H}_2 \\ -X\mathcal{H}_1X\mathcal{H}_2 + X\mathcal{H}_2X\mathcal{H}_1 & -X\mathcal{H}_1X\mathcal{H}_3 + X\mathcal{H}_2X\mathcal{H}_2 \end{pmatrix}.$$

By Lemma 5, we have: $\mathcal{H}_iX\mathcal{H}_j = \mathcal{H}_jX\mathcal{H}_i$, therefore the off-diagonal blocks here vanish, while the diagonal blocks are equal. Moreover, the diagonal blocks are circulant matrices and therefore can be represented as

$$X\mathcal{H}_2X\mathcal{H}_2 - X\mathcal{H}_3X\mathcal{H}_1 = q_0(x)I + q_1(x)\mathcal{P} + \dots q_{n-1}(x)\mathcal{P}^{n-1}, \quad (44)$$

which finishes the proof. \square

Lemma 9. *Functions $q_\ell(x)$ from Lemma 8 satisfy*

$$\nabla q_\ell(x) = A^\ell \nabla q_0(x) \iff \nabla q_\ell = A \nabla q_{\ell-1}, \quad \ell = 1, \dots, n-1.$$

Proof. According to equation (44), we can write:

$$\begin{aligned} q_0(x) &= \frac{1}{n} \operatorname{tr}(X(u)\mathcal{H}_2X(u)\mathcal{H}_2 - X(u)\mathcal{H}_3X(u)\mathcal{H}_1), \\ q_\ell(x) &= \frac{1}{n} \operatorname{tr}(\mathcal{P}^{-\ell}X(u)\mathcal{H}_2X(u)\mathcal{H}_2 - \mathcal{P}^{-\ell}X(u)\mathcal{H}_3X(u)\mathcal{H}_1). \end{aligned}$$

We denote by $u, a, b, c \in \mathbb{R}^n$ the first columns of the matrices $X(u)$, \mathcal{H}_1 , \mathcal{H}_2 , \mathcal{H}_3 , respectively. Then the i -th column of the matrix $X(u)$ is $\mathcal{P}^i u$, the i -th column of the matrix $\mathcal{P}^{-\ell}X(u)$ is $\mathcal{P}^{i-\ell}u$, and the entry (i, j) of the circulant matrix $X(u)\mathcal{H}_1$, say, is equal to $\langle u, \mathcal{P}^{j-i}a \rangle$. Therefore,

$$\begin{aligned} q_0(x) &= \frac{1}{n} \sum_{i=0}^{n-1} \left(\langle u, \mathcal{P}^i b \rangle \langle u, \mathcal{P}^{-i} b \rangle - \langle u, \mathcal{P}^i c \rangle \langle u, \mathcal{P}^{-i} a \rangle \right), \\ q_\ell(x) &= \frac{1}{n} \sum_{i=0}^{n-1} \left(\langle \mathcal{P}^{-\ell} u, \mathcal{P}^i b \rangle \langle u, \mathcal{P}^{-i} b \rangle - \langle \mathcal{P}^{-\ell} u, \mathcal{P}^i c \rangle \langle u, \mathcal{P}^{-i} a \rangle \right). \end{aligned}$$

Now we directly compute (we write $\nabla_u q_\ell(x)$ for the first n components of the gradients, and remember that the last n components vanish: $\nabla_v q_\ell(x) = 0$):

$$\begin{aligned} \nabla_u q_0(x) &= \frac{1}{n} \sum_{i=0}^{n-1} \left(\langle u, \mathcal{P}^{-i} b \rangle \mathcal{P}^i b + \langle u, \mathcal{P}^i b \rangle \mathcal{P}^{-i} b - \langle u, \mathcal{P}^{-i} a \rangle \mathcal{P}^i c - \langle u, \mathcal{P}^i c \rangle \mathcal{P}^{-i} a \right), \\ \nabla_u q_\ell(x) &= \frac{1}{n} \sum_{i=0}^{n-1} \left(\langle u, \mathcal{P}^{-i} b \rangle \mathcal{P}^{i+\ell} b + \langle u, \mathcal{P}^{i+\ell} b \rangle \mathcal{P}^{-i} b - \langle u, \mathcal{P}^{-i} a \rangle \mathcal{P}^{i+\ell} c - \langle u, \mathcal{P}^{i+\ell} c \rangle \mathcal{P}^{-i} a \right). \end{aligned}$$

Shifting in the second and the fourth sums on the right-hand side of the last equation index i by $-\ell$, we find:

$$\nabla_u q_\ell(x) = \sum_{i=0}^{n-1} \left(\langle u, \mathcal{P}^{-i} b \rangle \mathcal{P}^{i+\ell} b + \langle u, \mathcal{P}^i b \rangle \mathcal{P}^{-i+\ell} b - \langle u, \mathcal{P}^{-i} a \rangle \mathcal{P}^{i+\ell} c - \langle u, \mathcal{P}^i c \rangle \mathcal{P}^{-i+\ell} a \right).$$

Thus, we obtain $\nabla_u q_\ell(x) = \mathcal{P}^\ell \nabla_u q_0(x)$, which, of course, yields the required relation $\nabla q_\ell(x) = A^\ell \nabla q_0(x)$. \square

6. ASSOCIATED VECTOR FIELDS

Definition 10. *Let the matrix*

$$B = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

satisfy

$$B^2 = I.$$

Then the vector field

$$g(x) = J(x)B\nabla H_0(x) = B^T J(x)\nabla H_0(x) = B^T f_0(x)$$

is called associated to the vector field $f_0(x)$. The vector field $g(x)$ is Hamiltonian,

$$g(x) = J(x)\nabla K(x),$$

with the Hamilton function

$$K(x) = \alpha_0 H_0(x) + \alpha_1 H_1(x) + \dots + \alpha_{n-1} H_{n-1}(x).$$

This defines an equivalence relation on the set of vector fields $J(x)\nabla H(x)$ with the Hamilton functions $H(x)$ satisfying (22).

Lemma 11. *If vector field $g(x)$ is associated to $f_0(x)$ via the matrix B , then the following identities hold true:*

$$g'(x)g(x) = f'_0(x)f_0(x), \tag{45}$$

$$(g'(x))^2 = (f'_0(x))^2, \tag{46}$$

and

$$J(x)\nabla^2 H \cdot J(x)\nabla^2 K = J(x)\nabla^2 K \cdot J(x)\nabla^2 H, \tag{47}$$

$$(J(x)\nabla^2 H)^2 = (J(x)\nabla^2 K)^2, \tag{48}$$

$$(\nabla^2 H J(x))^2 = (\nabla^2 K J(x))^2. \tag{49}$$

Proof. We first check (45):

$$g'(x)g(x) = g'(x)B^T f_0(x) = B^T g'(x)f_0(x) = (B^T)^2 f'_0(x)f_0(x) = f'_0(x)f_0(x).$$

For (46) everything is similar:

$$g'(x)g'(x) = g'(x)B^T f'_0(x) = B^T g'(x)f'_0(x) = (B^T)^2 (f'_0(x))^2 = (f'_0(x))^2.$$

Further, we have, according to (20) and to (22):

$$J(x)\nabla^2 K = J(x)B \nabla^2 H = B^T J(x)\nabla^2 H = J(x)\nabla^2 H B^T.$$

These two formulas for $J(x)\nabla^2 K$, together with $(B^T)^2 = I$, yield (47) and (48). \square

Lemma 12. *In dimension $2n > 4$, there exist n linearly independent matrices*

$$B_j = \alpha_0^{(j)} I + \alpha_1^{(j)} A + \dots + \alpha_{n-1}^{(j)} A^{n-1} \quad (j = 0, \dots, n-1),$$

satisfying

$$B_j^2 = I, \tag{50}$$

and such that

$$\det(B_j - \lambda I) = (\lambda - 1)^{2n-2} (\lambda + 1)^2. \tag{51}$$

They are related by

$$\sum_{j=0}^{n-1} B_j = (n-2)I. \tag{52}$$

Proof. The matrix A from (29) has n eigenvalues $\lambda_0 = 1, \lambda_1 = \omega, \dots, \lambda_{n-1} = \omega^{n-1}$ with $\omega = \exp(2\pi i/n)$, all of the multiplicity 2, with two linearly independent eigenvectors. So, A is diagonalizable and its characteristic polynomial is of the form

$$\det(\lambda I - A) = (\lambda - 1)^2 (\lambda - \omega)^2 \dots (\lambda - \omega^{n-1})^2 = (\lambda^n - 1)^2.$$

We construct (according to the Lagrange interpolating formula) n polynomials of degree $n-1$,

$$B_j(\lambda) = \alpha_0^{(j)} + \alpha_1^{(j)} \lambda + \dots + \alpha_{n-1}^{(j)} \lambda^{n-1} \quad (j = 0, \dots, n-1)$$

such that

$$B_j(\lambda_j) = -1 \quad \text{and} \quad B_j(\lambda_k) = 1 \quad \text{for} \quad k \neq j. \tag{53}$$

Thus, each matrix $B_j = B_j(A)$ has $n-1$ double eigenvalues equal to 1 and one double eigenvalue equal to -1 , so that (51) is satisfied. As a corollary, each matrix B_j^2 has all $2n$ eigenvalues equal to 1, which, together with diagonalizability, yields (50). Finally, from (53) there follows that the degree $n-1$ polynomial $B_0(\lambda) + \dots + B_{n-1}(\lambda)$ takes the value $n-2$ at the n points $\lambda_0, \dots, \lambda_{n-1}$, therefore it is identically equal to $n-2$. This yields (52). Actually, the matrices B_j can be easily computed explicitly. The $n \times n$ diagonal blocks of B_j are given by

$$I - \frac{2}{n} \begin{pmatrix} 1 & \omega^j & \omega^{2j} & \dots & \omega^{(n-1)j} \\ \omega^{(n-1)j} & 1 & \omega^j & \dots & \omega^{(n-2)j} \\ \omega^{(n-2)j} & \omega^{(n-1)j} & 1 & \dots & \omega^{(n-3)j} \\ \dots & \dots & \dots & \dots & \dots \\ \omega^j & \omega^{2j} & \omega^{3j} & \dots & 1 \end{pmatrix}, \quad \omega = \exp(2\pi i/n),$$

so that $\alpha_0^{(j)} = 1 - \frac{2}{n}$ and $\alpha_k^{(j)} = -\frac{2}{n} \omega^{kj}$. □

Our main results are the following: for two associated vector fields f and g the Kahan maps Φ_f and Φ_g commute (Theorem 13), share n independent integrals of motion (Theorem 15), and share an invariant symplectic structure (Theorem 17).

7. COMMUTATIVITY OF MAPS

Theorem 13. *Let $f(x) = J(x)\nabla H(x)$ and $g(x) = J(x)\nabla K(x)$ be two associated vector fields, via the matrix B . Then the maps*

$$\Phi_f : x \mapsto \tilde{x} = (I - \varepsilon f'(x))^{-1} x = (I + \varepsilon f'(\tilde{x})) x, \quad (54)$$

$$\Phi_g : x \mapsto \hat{x} = (I - \varepsilon g'(x))^{-1} x = (I + \varepsilon g'(\hat{x})) x, \quad (55)$$

commute: $\Phi_f \circ \Phi_g = \Phi_g \circ \Phi_f$.

Proof. We have:

$$(\Phi_g \circ \Phi_f)(x) = (I - \varepsilon g'(\tilde{x}))^{-1} (I + \varepsilon f'(\tilde{x})) x, \quad (56)$$

and

$$(\Phi_f \circ \Phi_g)(x) = (I - \varepsilon f'(\hat{x}))^{-1} (I + \varepsilon g'(\hat{x})) x. \quad (57)$$

We prove the following matrix equation:

$$(I - \varepsilon g'(\tilde{x}))^{-1} (I + \varepsilon f'(\tilde{x})) = (I - \varepsilon f'(\hat{x}))^{-1} (I + \varepsilon g'(\hat{x})), \quad (58)$$

which is stronger than the vector equation $(\Phi_f \circ \Phi_g)(x) = (\Phi_g \circ \Phi_f)(x)$ expressing commutativity. Equation (58) is equivalent to

$$(I - \varepsilon f'(\hat{x})) (I - \varepsilon g'(\tilde{x}))^{-1} = (I + \varepsilon g'(\hat{x})) (I + \varepsilon f'(\tilde{x}))^{-1}. \quad (59)$$

From (46) we find:

$$\begin{aligned} (I - \varepsilon g'(\tilde{x}))^{-1} &= (I + \varepsilon g'(\tilde{x})) \left(I - \varepsilon^2 (f'(\tilde{x}))^2 \right)^{-1}, \\ (I + \varepsilon f'(\tilde{x}))^{-1} &= (I - \varepsilon f'(\tilde{x})) \left(I - \varepsilon^2 (f'(\tilde{x}))^2 \right)^{-1}. \end{aligned}$$

With this at hand, equation (59) is equivalent to

$$(I - \varepsilon f'(\hat{x})) (I + \varepsilon g'(\tilde{x})) = (I + \varepsilon g'(\hat{x})) (I - \varepsilon f'(\tilde{x})).$$

Here the quadratic in ε terms cancel by virtue of (40) and (33):

$$f'(\hat{x})g'(\tilde{x}) = f'(\hat{x})B^T f'(\tilde{x}) = B^T f'(\hat{x})f'(\tilde{x}) = g'(\hat{x})f'(\tilde{x}),$$

so that we are left with the terms linear in ε :

$$-f'(\hat{x}) + g'(\tilde{x}) = g'(\hat{x}) - f'(\tilde{x}). \quad (60)$$

Since the tensors f'' , g'' are constant, we have:

$$\begin{aligned} f'(\hat{x}) &= f'(x) + f''(\hat{x} - x) = f'(x) + 2\varepsilon f'' (I - \varepsilon g'(x))^{-1} g(x), \\ g'(\hat{x}) &= g'(x) + g''(\hat{x} - x) = g'(x) + 2\varepsilon g'' (I - \varepsilon g'(x))^{-1} g(x), \\ f'(\tilde{x}) &= f'(x) + f''(\tilde{x} - x) = f'(x) + 2\varepsilon f'' (I - \varepsilon f'(x))^{-1} f(x), \\ g'(\tilde{x}) &= g'(x) + g''(\tilde{x} - x) = g'(x) + 2\varepsilon g'' (I - \varepsilon f'(x))^{-1} f(x). \end{aligned}$$

Thus, equation (60) is equivalent to

$$\begin{aligned} f'' (I - \varepsilon g'(x))^{-1} g(x) + g'' (I - \varepsilon g'(x))^{-1} g(x) = \\ f'' (I - \varepsilon f'(x))^{-1} f(x) + g'' (I - \varepsilon f'(x))^{-1} f(x). \end{aligned} \quad (61)$$

At this point, we use the following statement.

Lemma 14. *For any vector $v \in \mathbb{C}^{2n}$ we have:*

$$g''(x)v = f''(x)(B^T v), \quad f''(x)v = g''(x)(B^T v). \quad (62)$$

We compute the matrices on the left-hand side of (61) with the help of (62), (39), (40):

$$\begin{aligned} f''(I - \varepsilon g'(x))^{-1} g(x) &= f''(I - \varepsilon^2(f'(x))^2)^{-1} (g(x) + \varepsilon g'(x)g(x)), \\ g''(I - \varepsilon g'(x))^{-1} g(x) &= f''(I - \varepsilon^2(f'(x))^2)^{-1} B^T (g(x) + \varepsilon g'(x)g(x)) \\ &= f''(I - \varepsilon^2(f'(x))^2)^{-1} (f(x) + \varepsilon f'(x)g(x)), \end{aligned}$$

and similarly

$$\begin{aligned} f''(I - \varepsilon f'(x))^{-1} f(x) &= f''(I - \varepsilon^2(f'(x))^2)^{-1} (f(x) + \varepsilon f'(x)f(x)) \\ g''(I - \varepsilon f'(x))^{-1} f(x) &= f''(I - \varepsilon^2(f'(x))^2)^{-1} B^T (f(x) + \varepsilon f'(x)f(x)) \\ &= f''(I - \varepsilon^2(f'(x))^2)^{-1} (g(x) + \varepsilon g'(x)f(x)). \end{aligned}$$

Collecting all the results and using (41) and (45), we see that the proof is complete. \square

Proof of Lemma 14. The identities in question are equivalent to

$$B^T(f''(x)v) = f''(x)(B^T v), \quad B^T(g''(x)v) = g''(x)(B^T v). \quad (63)$$

(Actually, both tensors f'' and g'' are constant, i.e., do not depend on x .) To prove the latter identities, we start with equation (33) written in components:

$$\sum_k (B^T)_{ik} \frac{\partial f_k}{\partial x_\ell} = \sum_k \frac{\partial f_i}{\partial x_k} (B^T)_{k\ell}.$$

Differentiating with respect to x_j , we get:

$$\sum_k (B^T)_{ik} \frac{\partial^2 f_k}{\partial x_j \partial x_\ell} = \sum_k \frac{\partial f_i}{\partial x_j \partial x_k} (B^T)_{k\ell}.$$

Hence,

$$\sum_{k,\ell} (B^T)_{ik} \frac{\partial^2 f_k}{\partial x_j \partial x_\ell} v_\ell = \sum_{k,\ell} \frac{\partial f_i}{\partial x_j \partial x_k} (B^T)_{k\ell} v_\ell,$$

which is nothing but the (i, j) entry of the matrix identity (63). \square

8. INTEGRALS OF MOTION

Theorem 15. *Let $f(x) = J(x)\nabla H(x)$ and $g(x) = J(x)\nabla K(x)$ be two associated vector fields, via the matrix B . Then the maps Φ_f and Φ_g share two functionally independent conserved quantities*

$$\tilde{H}(x, \varepsilon) = h(x, \Phi_f(x, \varepsilon), \varepsilon), \quad (64)$$

and

$$\tilde{K}(x, \varepsilon) = k(x, \Phi_g(x, \varepsilon), \varepsilon), \quad (65)$$

where

$$h(x, \tilde{x}, \varepsilon) = (2\varepsilon)^{-1} x^T J^{-1} \left(\frac{x + \tilde{x}}{2} \right) \tilde{x}, \quad (66)$$

and

$$k(x, \hat{x}, \varepsilon) = (2\varepsilon)^{-1} x^T J^{-1} \left(\frac{x + \hat{x}}{2} \right) \hat{x}. \quad (67)$$

Proof. First, we show that $\tilde{H}(x, \varepsilon)$ is an integral of motion for the map Φ_f . We start with giving several equivalent formulas for $\tilde{H}(x, \varepsilon)$. Upon using the skew-symmetry of $J(x)$ and formula (12), we can rewrite (64) as

$$\begin{aligned} \tilde{H}(x, \varepsilon) &= (2\varepsilon)^{-1} x^T J^{-1} \left(\frac{x + \tilde{x}}{2} \right) (\tilde{x} - x) \\ &= x^T J^{-1} \left(\frac{x + \tilde{x}}{2} \right) \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} J \left(\frac{x + \tilde{x}}{2} \right) \nabla H(x). \end{aligned}$$

Lemma 16. For any $x, y \in \mathbb{R}^n$, we have:

$$J(x) \nabla^2 H J(y) = J(y) \nabla^2 H J(x). \quad (68)$$

Proof. With notation (15), (30), we have to prove:

$$X \mathcal{H}_k Y = Y \mathcal{H}_k X, \quad k = 1, 2, 3.$$

But this follows directly from Lemma 5. □

From this lemma, there follows:

$$\left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} J \left(\frac{x + \tilde{x}}{2} \right) = J \left(\frac{x + \tilde{x}}{2} \right) \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1},$$

and

$$\tilde{H}(x, \varepsilon) = x^T \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1} \nabla H(x).$$

Expanding into a power series in ε , we find:

$$\begin{aligned} \tilde{H}(x, \varepsilon) &= x^T \sum_{k=0}^{\infty} \varepsilon^k \left(\nabla^2 H J(x) \cdots \nabla^2 H J(x) \right) \nabla H(x) \\ &= x^T \sum_{k=0}^{\infty} \varepsilon^k \left(\nabla^2 H J(x) \cdots \nabla^2 H J(x) \nabla^2 H \right) x. \end{aligned}$$

The matrix in the parentheses involves $k + 1$ times $\nabla^2 H$ and k times $J(x)$, therefore it is symmetric if k is even, and skew-symmetric if k is odd. Therefore, all terms with odd k vanish. We have the following equivalent expressions:

$$\tilde{H}(x, \varepsilon) = x^T (\nabla^2 H) \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} x \quad (69)$$

$$= x^T (\nabla^2 H) \left(I - \varepsilon^2 (J(x) \nabla^2 H)^2 \right)^{-1} x \quad (70)$$

$$= x^T \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1} (\nabla^2 H) x \quad (71)$$

$$= x^T \left(I - \varepsilon^2 (\nabla^2 H J(x))^2 \right)^{-1} (\nabla^2 H) x. \quad (72)$$

Moreover, in (69) and (71) one can replace ε by $-\varepsilon$.

The fact that $\tilde{H}(x, \varepsilon)$ is an even function of ε ensures that it is an integral of Φ_f . Indeed, by virtue of (4) we have:

$$\tilde{H}(x, -\varepsilon) = h(x, \Phi_f(x, -\varepsilon), -\varepsilon) = h(x, \Phi_f^{-1}(x, \varepsilon), -\varepsilon) = h(\Phi_f^{-1}(x, \varepsilon), x, \varepsilon).$$

The last equality follows from the property of the function h ,

$$h(x, y, \varepsilon) = h(y, x, -\varepsilon),$$

which follows from the definition (66) by the skew-symmetry of the matrix J . Thus, if $\tilde{H}(x, \varepsilon) = \tilde{H}(x, -\varepsilon)$, then

$$\tilde{H}(x, \varepsilon) = \tilde{H}(\Phi_f^{-1}(x, \varepsilon), \varepsilon),$$

which proves the claim.

Next, we show that $\tilde{K}(x, \varepsilon)$ also is an integral of motion for the map Φ_f . For this goal, we first compute, based on (72):

$$\tilde{K}(\tilde{x}, \varepsilon) = \tilde{x}^T \left(I - \varepsilon^2 (\nabla^2 K J(\tilde{x}))^2 \right)^{-1} (\nabla^2 K) \tilde{x}.$$

By virtue of (13) we have:

$$\begin{aligned} \tilde{K}(\tilde{x}, \varepsilon) &= x^T \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right)^T \left(I - \varepsilon J(x) \nabla^2 H \right)^{-T} \left(I - \varepsilon^2 (\nabla^2 K J(\tilde{x}))^2 \right)^{-1} \times \\ &\quad \times (\nabla^2 K) \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right) x. \end{aligned}$$

By Lemma 16 we have:

$$\begin{aligned} \tilde{K}(\tilde{x}, \varepsilon) &= x^T \left(I + \varepsilon \nabla^2 H J(x) \right)^{-1} \left(I - \varepsilon \nabla^2 H J(\tilde{x}) \right) \left(I - \varepsilon^2 (\nabla^2 K J(\tilde{x}))^2 \right)^{-1} \times \\ &\quad \times (\nabla^2 K) \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right) \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} x. \end{aligned}$$

Next, we find:

$$\begin{aligned} (\nabla^2 K) \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right) &= B(\nabla^2 H) \left(I + \varepsilon J(\tilde{x}) \nabla^2 H \right) \\ &= B \left(I + \varepsilon \nabla^2 H J(\tilde{x}) \right) (\nabla^2 H) \\ &= \left(I + \varepsilon \nabla^2 H J(\tilde{x}) \right) B(\nabla^2 H) \\ &= \left(I + \varepsilon \nabla^2 H J(\tilde{x}) \right) (\nabla^2 K). \end{aligned}$$

Here, the last but one equality is justified as follows:

$$B \nabla^2 H J(\tilde{x}) = \nabla^2 H B^T J(\tilde{x}) = \nabla^2 H J(\tilde{x}) B.$$

Similarly, we find:

$$(\nabla^2 K) \left(I - \varepsilon J(x) \nabla^2 H \right)^{-1} = \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1} (\nabla^2 K).$$

Collecting all the results, we have:

$$\begin{aligned}\tilde{K}(\tilde{x}, \varepsilon) &= x^T \left(I + \varepsilon \nabla^2 H J(x) \right)^{-1} \left(I - \varepsilon \nabla^2 H J(\tilde{x}) \right) \left(I - \varepsilon^2 (\nabla^2 K J(\tilde{x}))^2 \right)^{-1} \times \\ &\quad \times \left(I + \varepsilon \nabla^2 H J(\tilde{x}) \right) \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1} (\nabla^2 K) x.\end{aligned}$$

Applying equation (49) twice, we find:

$$\begin{aligned}\tilde{K}(\tilde{x}, \varepsilon) &= x^T \left(I + \varepsilon \nabla^2 H J(x) \right)^{-1} \left(I - \varepsilon \nabla^2 H J(\tilde{x}) \right) \left(I - \varepsilon^2 (\nabla^2 H J(\tilde{x}))^2 \right)^{-1} \times \\ &\quad \times \left(I + \varepsilon \nabla^2 H J(\tilde{x}) \right) \left(I - \varepsilon \nabla^2 H J(x) \right)^{-1} (\nabla^2 K) x \\ &= x^T \left(I - \varepsilon^2 (\nabla^2 H J(x))^2 \right)^{-1} (\nabla^2 K) x \\ &= x^T \left(I - \varepsilon^2 (\nabla^2 K J(x))^2 \right)^{-1} (\nabla^2 K) x \\ &= \tilde{K}(x, \varepsilon),\end{aligned}$$

which finishes the proof. \square

9. INVARIANT POISSON STRUCTURE

Theorem 17. *Let $f(x) = J(x) \nabla H(x)$ and $g(x) = J(x) \nabla K(x)$ be two associated vector fields, via the matrix B . Then both maps Φ_f and Φ_g are Poisson with respect to the brackets with the Poisson tensor $\Pi(x)$ given by*

$$\Pi(x) = J(x) - \varepsilon^2 J(x) \cdot \nabla^2 H \cdot J(x) \cdot \nabla^2 H \cdot J(x). \quad (73)$$

This theorem is a direct consequence of the following two statements combined with Lemma 9.

Proposition 18. *For the matrix $\Pi(x)$ from (73), we have:*

$$d\Phi_f(x) \Pi(x) (d\Phi_f(x))^T = \Pi(\tilde{x}). \quad (74)$$

Proposition 19. *A matrix*

$$\Pi(x) = (1 - \varepsilon^2 q_0(x)) J(x) - \varepsilon^2 \sum_{\ell=1}^{n-1} q_\ell(x) A^\ell J(x) \quad (75)$$

is a Poisson tensor if and only if the functions $q_\ell(x)$ satisfy

$$\nabla q_\ell(x) = A \nabla q_{\ell-1}(x), \quad \ell \in \mathbb{Z}/(n\mathbb{Z}), \quad (76)$$

or, equivalently,

$$\frac{\partial q_\ell}{\partial x_i} = \frac{\partial q_{\ell-1}}{\partial x_{i+1}}, \quad \ell \in \mathbb{Z}/(n\mathbb{Z}), \quad i \in [1, 2n], \quad (77)$$

where the latter equation for $i = n$ and for $i = 2n$ should be read as

$$\frac{\partial q_\ell}{\partial x_n} = \frac{\partial q_{\ell-1}}{\partial x_1}, \quad \text{resp.} \quad \frac{\partial q_\ell}{\partial x_{2n}} = \frac{\partial q_{\ell-1}}{\partial x_{n+1}}.$$

Proof of Proposition 18. With expression (8) for $d\Phi_f(x)$, equation (74) turns into

$$(I + \varepsilon f'(\tilde{x}))\Pi(x)(I + \varepsilon f'(\tilde{x}))^T = (I - \varepsilon f'(x))\Pi(\tilde{x})(I - \varepsilon f'(x))^T. \quad (78)$$

We have: $\Pi(x) = (I - \varepsilon^2(J(x)\nabla^2 H)^2)J(x)$. According to Lemma 8, $I - \varepsilon^2(J(x)\nabla^2 H)^2$ is a matrix polynomial of A . By virtue of (33), this matrix commutes with $f'(\tilde{x})$ (actually, with f' evaluated at any point). Therefore, equation (78) is equivalent to

$$\begin{aligned} & (I - \varepsilon^2(J(x)\nabla^2 H)^2)(I + \varepsilon f'(\tilde{x}))J(x)(I + \varepsilon f'(\tilde{x}))^T \\ &= (I - \varepsilon^2(J(\tilde{x})\nabla^2 H)^2)(I - \varepsilon f'(x))J(\tilde{x})(I - \varepsilon f'(x))^T. \end{aligned} \quad (79)$$

Lemma 20. *We have:*

$$I - \varepsilon f'(x) = (I - \varepsilon J(x)\nabla^2 H) \begin{pmatrix} XX_1^{-1} & 0 \\ -X_2X_1^{-1} & I \end{pmatrix}, \quad (80)$$

and

$$I + \varepsilon f'(\tilde{x}) = (I + \varepsilon J(\tilde{x})\nabla^2 H) \begin{pmatrix} \tilde{X}X_1^{-1} & 0 \\ X_2X_1^{-1} & I \end{pmatrix}, \quad (81)$$

where

$$X = X(u), \quad \tilde{X} = X(\tilde{u}), \quad X_1 = X\left(\frac{u + \tilde{u}}{2}\right), \quad X_2 = X\left(\frac{\tilde{v} - v}{2}\right). \quad (82)$$

With this lemma and Lemma 16, according to which matrices $J(x)\nabla^2 H$ and $J(\tilde{x})\nabla^2 H$ commute, we can rewrite (79) as

$$\begin{aligned} & (I + \varepsilon J(x)\nabla^2 H) \begin{pmatrix} \tilde{X}X_1^{-1} & 0 \\ X_2X_1^{-1} & I \end{pmatrix} J(x) \begin{pmatrix} X_1^{-1}\tilde{X} & X_1^{-1}X_2 \\ 0 & I \end{pmatrix} (I - \varepsilon \nabla^2 H J(\tilde{x})) \\ &= (I - \varepsilon J(\tilde{x})\nabla^2 H) \begin{pmatrix} XX_1^{-1} & 0 \\ -X_2X_1^{-1} & I \end{pmatrix} J(\tilde{x}) \begin{pmatrix} X_1^{-1}X & -X_1^{-1}X_2 \\ 0 & I \end{pmatrix} (I + \varepsilon \nabla^2 H J(x)). \end{aligned} \quad (83)$$

In the following computation, we repeatedly use the property of cyclic Hankel matrices formulated in Lemma 5. We compute:

$$\begin{aligned} & \begin{pmatrix} \tilde{X}X_1^{-1} & 0 \\ X_2X_1^{-1} & I \end{pmatrix} J(x) \begin{pmatrix} X_1^{-1}\tilde{X} & X_1^{-1}X_2 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & \tilde{X}X_1^{-1}X \\ -XX_1^{-1}\tilde{X} & -XX_1^{-1}X_2 + X_2X_1^{-1}X \end{pmatrix} \\ &= \begin{pmatrix} 0 & \tilde{X}X_1^{-1}X \\ -XX_1^{-1}\tilde{X} & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}, \end{aligned}$$

where

$$Y = \tilde{X}X_1^{-1}X = XX_1^{-1}\tilde{X}$$

is a cyclic Hankel matrix (according to Lemma 5). Similarly,

$$\begin{aligned} & \begin{pmatrix} XX_1^{-1} & 0 \\ -X_2X_1^{-1} & I \end{pmatrix} J(\tilde{x}) \begin{pmatrix} X_1^{-1}X & -X_1^{-1}X_2 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & XX_1^{-1}\tilde{X} \\ -\tilde{X}X_1^{-1}X & -\tilde{X}X_1^{-1}X_2 + X_2X_1^{-1}\tilde{X} \end{pmatrix} \\ &= \begin{pmatrix} 0 & XX_1^{-1}\tilde{X} \\ -\tilde{X}X_1^{-1}X & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}. \end{aligned}$$

Observe that

$$\begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} = J(x)J^{-1}\left(\frac{x+\tilde{x}}{2}\right)J(\tilde{x}) = J(\tilde{x})J^{-1}\left(\frac{x+\tilde{x}}{2}\right)J(x).$$

Now, equation (83) takes the form

$$\begin{aligned} & (I + \varepsilon J(x)\nabla^2 H) \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} (I - \varepsilon \nabla^2 H J(\tilde{x})) \\ &= (I - \varepsilon J(\tilde{x})\nabla^2 H) \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} (I + \varepsilon \nabla^2 H J(x)), \end{aligned} \quad (84)$$

which is obviously true due to properties of cyclic Hankel matrices. \square

Proof of Lemma 20. Observe that (81) is obtained from (80) upon replacing ε by $-\varepsilon$ (which is equivalent to replacing \tilde{x} by \underline{x}) with a subsequent shift in time. Therefore, it is sufficient to prove (80). The latter formula can be equivalently rewritten as

$$(I - \varepsilon J(x)\nabla^2 H)^{-1}(I - \varepsilon f'(x)) = \begin{pmatrix} XX_1^{-1} & 0 \\ -X_2X_1^{-1} & I \end{pmatrix},$$

or

$$(I - \varepsilon J(x)\nabla^2 H)^{-1}(I - \varepsilon f'(x))J\left(\frac{x+\tilde{x}}{2}\right) = \begin{pmatrix} 0 & X \\ -X_1 & -X_2 \end{pmatrix},$$

or

$$\left((I - \varepsilon J(x)\nabla^2 H)^{-1}(I - \varepsilon f'(x)) - I\right)J\left(\frac{x+\tilde{x}}{2}\right) = \begin{pmatrix} 0 & X - X_1 \\ 0 & -X_2 \end{pmatrix}.$$

Taking into account equation (10) and definitions (82), we put the latter equation into the form

$$\varepsilon(I - \varepsilon J(x)\nabla^2 H)^{-1}J'\nabla H(x)J\left(\frac{x+\tilde{x}}{2}\right) = \begin{pmatrix} 0 & X(\frac{\tilde{u}-u}{2}) \\ 0 & X(\frac{\tilde{v}-v}{2}) \end{pmatrix}.$$

It remains to observe that, according to (34), (37), the latter equation is equivalent to (consists of n cyclically shifted versions of)

$$\varepsilon(I - \varepsilon J(x)\nabla^2 H)^{-1}J\left(\frac{x+\tilde{x}}{2}\right)\nabla H(x) = \begin{pmatrix} \frac{\tilde{u}-u}{2} \\ \frac{\tilde{v}-v}{2} \end{pmatrix} = \frac{\tilde{x}-x}{2},$$

which is nothing but (12). \square

Proof of Proposition 19. Like in Section 4, we have to check the Jacobi identity (17). Again, due to the block-diagonal structure of the matrix $\Pi(x)$, one only has to check this for the

cases $i, j \in \{1, \dots, n\}, k \in \{n+1, \dots, 2n\}$ and $i, j \in \{n+1, \dots, 2n\}, k \in \{1, \dots, n\}$, where (17) simplifies to (18). We compute:

$$\{x_j, x_k\} = (1 - \varepsilon^2 q_0(x)) J_{jk}(x) - \varepsilon^2 \sum_{\ell=1}^{n-1} q_\ell(x) (A^\ell J(x))_{jk}.$$

Taking into account that $J(x)$ only depends on x_m with $1 \leq m \leq n$, we have:

$$\begin{aligned} \{x_i, \{x_j, x_k\}\} &= (1 - \varepsilon^2 q_0(x)) \sum_{m=1}^n \frac{\partial J_{jk}(x)}{\partial x_m} \{x_i, x_m\} - \varepsilon^2 \sum_{\ell=1}^{n-1} q_\ell(x) \sum_{m=1}^n \frac{\partial (A^\ell J(x))_{jk}}{\partial x_m} \{x_i, x_m\} \\ &\quad - \varepsilon^2 \sum_{\ell=0}^{n-1} \sum_{m=1}^{2n} \frac{\partial q_\ell(x)}{\partial x_m} (A^\ell J(x))_{jk} \{x_i, x_m\}. \end{aligned} \quad (85)$$

We first deal with the terms from the first line. They can be represented as

$$\begin{aligned} &(1 - \varepsilon^2 q_0(x))^2 \sum_{m=1}^n \frac{\partial J_{jk}(x)}{\partial x_m} J_{im}(x) \\ &- \varepsilon^2 \sum_{p=1}^{n-1} (1 - \varepsilon^2 q_0(x)) q_p(x) \sum_{m=1}^n \frac{\partial J_{jk}(x)}{\partial x_m} (A^p J(x))_{im} \\ &- \varepsilon^2 \sum_{\ell=1}^{n-1} q_\ell(x) (1 - \varepsilon^2 q_0(x)) \sum_{m=1}^n \frac{\partial (A^\ell J(x))_{jk}}{\partial x_m} J_{im}(x) \\ &+ \varepsilon^4 \sum_{\ell=1}^{n-1} \sum_{p=1}^{n-1} q_\ell(x) q_p(x) \sum_{m=1}^n \frac{\partial (A^\ell J(x))_{jk}}{\partial x_m} (A^p J(x))_{im}. \end{aligned} \quad (86)$$

The contribution of these terms to (18) vanishes due to the following computation:

$$\begin{aligned} \sum_{m=1}^n \frac{\partial (A^\ell J(x))_{jk}}{\partial x_m} (A^p J(x))_{im} &= - \sum_{m=1}^n \frac{\partial x_{j+k+\ell-1 \pmod n}}{\partial x_m} x_{i+m+p-1 \pmod n} \\ &= - \sum_{m=1}^n \delta_{m, j+k+\ell-1 \pmod n} x_{i+m+p-1 \pmod n} \\ &= - x_{i+j+k+\ell+p-2 \pmod n}. \end{aligned}$$

This expression is symmetric with respect to $i \leftrightarrow j$, which results in a zero contribution to (18). Note this result for $\ell = p = 0$ is equivalent to the Jacobi identity for the bracket $\{\cdot, \cdot\}_J$, while for $\ell = p \neq 0$ it is equivalent to the Jacobi identity for the bracket $\{\cdot, \cdot\}_{A^\ell J}$. The general result is equivalent to the compatibility of the brackets $\{\cdot, \cdot\}_{A^\ell J}$ and $\{\cdot, \cdot\}_{A^p J}$.

The terms from the second line in (85) can be represented as

$$\begin{aligned} &- \varepsilon^2 \sum_{\ell=0}^{n-1} \sum_{m=1}^{2n} \frac{\partial q_\ell(x)}{\partial x_m} (A^\ell J(x))_{jk} J_{im}(x) \\ &+ \varepsilon^4 \sum_{\ell=0}^{n-1} \sum_{p=0}^{n-1} \sum_{m=1}^{2n} \frac{\partial q_\ell(x)}{\partial x_m} q_p(x) (A^\ell J(x))_{jk} (A^p J(x))_{im}. \end{aligned} \quad (87)$$

Due to the block structure of the matrix $J(x)$, if $i, j \in \{1, \dots, n\}$, one can restrict the summation index m here to the range $m \in \{n+1, \dots, 2n\}$, while if $i, j \in \{n+1, \dots, 2n\}$,

one can restrict the summation index m to the range $m \in \{1, \dots, n\}$. Both possibilities are considered analogously, therefore we concentrate on the second one. Thus, assume that $i, j \in \{n+1, \dots, 2n\}$. Then the terms in (87) quadratic in ε can be transformed as follows:

$$\begin{aligned} & -\varepsilon^2 \sum_{\ell=0}^{n-1} \sum_{m=1}^n \frac{\partial q_\ell(x)}{\partial x_m} (A^\ell J(x))_{jk} J_{im}(x) \\ &= -\varepsilon^2 \sum_{\ell=0}^{n-1} \sum_{m=1}^n \frac{\partial q_\ell(x)}{\partial x_m} x_{j+k+\ell-1 \pmod{n}} x_{i+m-1 \pmod{n}} \\ &= -\varepsilon^2 \sum_{a=1}^n \sum_{b=1}^n \frac{\partial q_{a-j-k+1 \pmod{n}}(x)}{\partial x_{b-i+1 \pmod{n}}} x_a x_b. \end{aligned}$$

Similarly, the terms in (87) of degree 4 in ε are transformed as follows:

$$\begin{aligned} & \varepsilon^4 \sum_{\ell=0}^{n-1} \sum_{p=0}^{n-1} \sum_{m=1}^n \frac{\partial q_\ell(x)}{\partial x_m} q_p(x) (A^\ell J(x))_{jk} (A^p J(x))_{im} \\ &= \varepsilon^4 \sum_{\ell=0}^{n-1} \sum_{p=0}^{n-1} \sum_{m=1}^n \frac{\partial q_\ell(x)}{\partial x_m} q_p(x) x_{j+k+\ell-1 \pmod{n}} x_{i+m+p-1 \pmod{n}} \\ &= \varepsilon^4 \sum_{a=1}^n \sum_{b=1}^n \sum_{p=0}^{n-1} \frac{\partial q_{a-j-k+1 \pmod{n}}(x)}{\partial x_{b-i-p+1 \pmod{n}}} q_p(x) x_a x_b. \end{aligned}$$

Thus, we arrive at the following expressions for the quantities π_{ijk} in (18) in the case $i, j \in \{n+1, \dots, 2n\}$ and $k \in \{1, \dots, n\}$:

$$\begin{aligned} \pi_{ijk} &= -\varepsilon^2 \sum_{a=1}^n \sum_{b=1}^n \left(\frac{\partial q_{a-j-k+1 \pmod{n}}(x)}{\partial x_{b-i+1 \pmod{n}}} - \frac{\partial q_{a-i-k+1 \pmod{n}}(x)}{\partial x_{b-j+1 \pmod{n}}} \right) x_a x_b \\ &+ \varepsilon^4 \sum_{a=1}^n \sum_{b=1}^n \sum_{p=0}^{n-1} \left(\frac{\partial q_{a-j-k+1 \pmod{n}}(x)}{\partial x_{b-i-p+1 \pmod{n}}} - \frac{\partial q_{a-i-k+1 \pmod{n}}(x)}{\partial x_{b-j-p+1 \pmod{n}}} \right) q_p(x) x_a x_b. \quad (88) \end{aligned}$$

We mention that in the case $i, j \in \{1, \dots, n\}$ and $k \in \{n+1, \dots, 2n\}$, the expression for π_{ijk} is almost literally the same, but with the indices $b-i+1 \pmod{n}$ etc. replaced by their \pmod{n} representatives in the interval $[n+1, 2n]$.

It remains to observe that (88) is equal to zero by virtue of (77). \square

10. DIFFERENTIAL EQUATIONS FOR THE CONSERVED QUANTITIES OF MAPS Φ_f, Φ_g

Theorem 21. *Let $f(x) = J(x) \nabla H_0(x)$ and $g(x) = J(x) \nabla K_0(x)$ be two associated vector fields, via the matrix B . Then the rational functions $\tilde{H}_0(x, \varepsilon)$ and $\tilde{K}_0(x, \varepsilon)$ are related by the same first order differential equation as the quadratic polynomials $H_0(x)$ and $K_0(x)$:*

$$\nabla \tilde{K}_0(x, \varepsilon) = B \nabla \tilde{H}_0(x, \varepsilon). \quad (89)$$

As a consequence, they satisfy the same second order differential equation (22) as the polynomials $H_0(x)$ and $K_0(x)$.

Proof. We start the proof with the derivation of a convenient formula for $\tilde{H}_0(x, \varepsilon)$. From (72) we have:

$$\tilde{H}_0(x, \varepsilon) = x^T \left(I - \varepsilon^2 (\nabla^2 H_0 J(x))^2 \right)^{-1} \nabla H_0(x). \quad (90)$$

Lemma 22. *We have:*

$$\left(I - \varepsilon^2 (\nabla^2 H_0 J(x))^2 \right)^{-1} = \sum_{i=0}^{n-1} r_i(x, \varepsilon) A^i, \quad (91)$$

where the functions $r_i(x, \varepsilon)$ satisfy differential equations

$$\nabla r_{i-1}(x, \varepsilon) = A \nabla r_i(x, \varepsilon), \quad i = 1, \dots, n-1. \quad (92)$$

From (90) and (91), we find:

$$\begin{aligned} \tilde{H}_0(x, \varepsilon) &= \sum_{i=0}^{n-1} r_i(x, \varepsilon) x^T A^i \nabla H_0(x) \\ &= \sum_{i=0}^{n-1} r_i(x, \varepsilon) x^T \nabla H_i(x) \\ &= 2 \sum_{i=0}^{n-1} r_i(x, \varepsilon) H_i(x). \end{aligned} \quad (93)$$

Differentiate formula (93), taking into account differential equations $A \nabla H_{i-1} = \nabla H_i$ and $A \nabla r_i = \nabla r_{i-1}$. We have:

$$\begin{aligned} A \nabla \tilde{H}_0(x, \varepsilon) &= 2 \sum_{i=0}^{n-1} (r_i(x, \varepsilon) A \nabla H_i(x) + H_i(x) A \nabla r_i(x, \varepsilon)) \\ &= 2 \sum_{i=0}^{n-1} (r_i(x, \varepsilon) \nabla H_{i+1}(x) + H_i(x) \nabla r_{i-1}(x, \varepsilon)) \\ &= \nabla \left(2 \sum_{i=0}^{n-1} r_i(x, \varepsilon) H_{i+1}(x) \right). \end{aligned} \quad (94)$$

By induction, we find:

$$A^m \nabla \tilde{H}_0(x, \varepsilon) = \nabla \left(2 \sum_{i=0}^{n-1} r_i(x, \varepsilon) H_{i+m}(x) \right), \quad m = 0, 1, \dots, n-1. \quad (95)$$

For any matrix polynomial $B = \beta_0 I + \beta_1 A + \dots + \beta_{n-1} A^{n-1}$, the Hamilton function $K_0(x)$ of the corresponding vector field is defined by $\nabla K_0(x) = B \nabla H_0(x)$, and we have: $K_0(x) = \beta_0 H_0(x) + \beta_1 H_1(x) + \dots + \beta_{n-1} H_{n-1}(x)$. As a consequence of the commutativity of B and A , for the functions $K_i(x)$ defined by $\nabla K_i(x) = A^i \nabla K_0(x)$, we also have: $K_i(x) = \beta_0 H_i(x) + \beta_1 H_{i+1}(x) + \dots + \beta_{n-1} H_{i+n-1}(x)$. Therefore, we derive from (95):

$$B \nabla \tilde{H}_0(x, \varepsilon) = \nabla \left(2 \sum_{i=0}^{n-1} r_i(x, \varepsilon) K_i(x) \right). \quad (96)$$

If now the vector fields with the Hamilton functions $H_0(x)$ and $K_0(x)$ are associated, that is, if $B^2 = I$, then, according to (49), we have:

$$\left(I - \varepsilon^2(\nabla^2 K_0 J(x))^2\right)^{-1} = \left(I - \varepsilon^2(\nabla^2 H_0 J(x))^2\right)^{-1} = \sum_{i=0}^{n-1} r_i(x, \varepsilon) A^i,$$

so that

$$\tilde{K}_0(x, \varepsilon) = 2 \sum_{i=1}^{n-1} r_i(x, \varepsilon) K_i(x).$$

Comparing this with (96), we arrive at equation (89). \square

Corollary 23. *The functions $\tilde{H}_0(x, \varepsilon)$ and $\tilde{K}_0(x, \varepsilon)$ are in involution with respect to both Poisson brackets, the original one with the Poisson tensor $J(x)$ and the perturbed one with the Poisson tensor $\Pi(x)$.*

Proof. The first statement follows directly from (89). For the second statement, we compute, according to (73) and (43):

$$\begin{aligned} \{\tilde{H}_0(x, \varepsilon), \tilde{K}_0(x, \varepsilon)\}_{\Pi} &= (\nabla \tilde{H}_0)^T \Pi(x) \nabla \tilde{K}_0 \\ &= (\nabla \tilde{H}_0)^T \left(I - \varepsilon^2(J(x) \nabla^2 H_0)^2\right) J(x) \nabla \tilde{K}_0 \\ &= (\nabla \tilde{H}_0)^T \left(I - \varepsilon^2 \sum_{i=0}^{n-1} q_i(x) A^i\right) J(x) \nabla \tilde{K}_0 \\ &= (\nabla \tilde{H}_0)^T \left(I - \varepsilon^2 \sum_{i=0}^{n-1} q_i(x) A^i B^T\right) J(x) \nabla \tilde{H}_0 = 0, \end{aligned}$$

because the (diagonal blocks of the) matrices $A^i B^T$ are circulant matrices. \square

Proof of Lemma 22. We have:

$$I - \varepsilon^2(\nabla^2 H_0 J(x))^2 = \left(I - \varepsilon^2(J(x) \nabla^2 H_0)^2\right)^T = I - \varepsilon^2 \sum_{i=0}^{n-1} q_i(x) A^{-i}.$$

Since the inverse of a circulant matrix is also circulant, we have:

$$\left(I - \varepsilon^2 \sum_{i=1}^{n-1} q_i(x) A^{-i}\right)^{-1} = \sum_{i=0}^{n-1} r_i(x, \varepsilon) A^i. \quad (97)$$

The coefficients $r_i(x)$ are determined just from the first column of the matrix identity obtained by left multiplying the right-hand side of the previous equation by the inverse of the left-hand side:

$$\begin{pmatrix} 1 - \varepsilon^2 q_0 & -\varepsilon^2 q_1 & -\varepsilon^2 q_2 & \dots & -\varepsilon^2 q_{n-1} \\ -\varepsilon^2 q_{n-1} & 1 - \varepsilon^2 q_0 & -\varepsilon^2 q_1 & \dots & -\varepsilon^2 q_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ -\varepsilon^2 q_1 & -\varepsilon^2 q_2 & -\varepsilon^2 q_3 & \dots & 1 - \varepsilon^2 q_0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \dots \\ r_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

A straightforward check shows that the unique solution of this system is given by

$$r_k(x, \varepsilon) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{\omega^{kj}}{s_j(x, \varepsilon)}, \quad k = 0, \dots, n-1, \quad (98)$$

where

$$s_j(x, \varepsilon) = 1 - \varepsilon^2 \sum_{m=0}^{n-1} \omega^{jm} q_m(x), \quad \omega = \exp(2\pi i/n).$$

It remains to show that functions (98) satisfy differential equations (92). We compute:

$$\nabla r_k = \frac{\varepsilon^2}{n} \sum_{j=0}^{n-1} \frac{\omega^{kj}}{s_j^2} \sum_{m=0}^{n-1} \omega^{jm} \nabla q_m = \frac{\varepsilon^2}{n} \sum_{j=0}^{n-1} \frac{1}{s_j^2} \left(\sum_{m=0}^{n-1} \omega^{j(k+m)} A^m \right) \nabla q_0.$$

Now (92) follows from the obvious relation

$$A \left(\sum_{m=0}^{n-1} \omega^{j(k+m)} A^m \right) = \sum_{m=0}^{n-1} \omega^{j(k+m-1)} A^m. \quad \square$$

11. EXAMPLES

Dimension $2n = 4$. Here, we are dealing with the following Lie-Poisson tensor:

$$J(x) = \begin{pmatrix} 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_2 & x_1 \\ -x_1 & -x_2 & 0 & 0 \\ -x_2 & -x_1 & 0 & 0 \end{pmatrix}.$$

In coordinates, the non-vanishing Poisson brackets are

$$\begin{aligned} \{x_1, x_3\} &= \{x_2, x_4\} = x_1, \\ \{x_1, x_4\} &= \{x_2, x_3\} = x_2 \end{aligned}$$

(and those being obtained from these ones by skew-symmetry).

General solution of equation (20) with $n = 2$ is

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & 0 & 0 \\ \alpha_1 & \alpha_0 & 0 & 0 \\ 0 & 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \alpha_1 & \alpha_0 \end{pmatrix}, \quad (99)$$

and the corresponding set of functions in involution is generated by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (100)$$

Since $A^2 = I$, we can take $B = A$ for the associated vector field in this case. Note also that $A = A^T$, a peculiarity of the case $2n = 4$.

The Hesse matrices \mathcal{H} of admissible Hamilton functions $H_0(x) = H(x)$ satisfying (22) are of the form

$$\mathcal{H} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & h_1 & h_4 & h_3 \\ h_3 & h_4 & h_5 & h_6 \\ h_4 & h_3 & h_6 & h_5 \end{pmatrix}. \quad (101)$$

The Hesse matrix of the commuting Hamilton function $H_1(x) = K(x)$ is obtained by the simultaneous flips of the pairs $a = (h_1, h_2)$, $b = (h_3, h_4)$, and $c = (h_5, h_6)$.

Lemma 8 in the present case of dimension $2n = 4$ gives the following results:

$$(J(x)\nabla^2 H)^2 = q_0(x)I + q_1(x)A = \begin{pmatrix} q_0(x) & q_1(x) & 0 & 0 \\ q_1(x) & q_0(x) & 0 & 0 \\ 0 & 0 & q_0(x) & q_1(x) \\ 0 & 0 & q_1(x) & q_0(x) \end{pmatrix}, \quad (102)$$

where

$$q_0(x) = \frac{1}{4}\text{tr}((J(x)\nabla^2 H)^2) = \alpha(x_1^2 + x_2^2) + 2\beta x_1 x_2, \quad (103)$$

$$q_1(x) = \frac{1}{4}\text{tr}(A(J(x)\nabla^2 H)^2) = \beta(x_1^2 + x_2^2) + 2\alpha x_1 x_2, \quad (104)$$

with

$$\alpha = h_3^2 + h_4^2 - h_1 h_5 - h_2 h_6, \quad \beta = 2h_3 h_4 - h_1 h_6 - h_2 h_5.$$

Functions $q_0(x), q_1(x)$ satisfy

$$\nabla q_1(x) = A\nabla q_0(x). \quad (105)$$

This follows from Lemma 9, but is also obvious from the explicit formulas (103), (104).

Dimension $2n = 6$. Here, we are dealing with the following Lie-Poisson tensor:

$$J(x) = \begin{pmatrix} 0 & 0 & 0 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & x_2 & x_3 & x_1 \\ 0 & 0 & 0 & x_3 & x_1 & x_2 \\ -x_1 & -x_2 & -x_3 & 0 & 0 & 0 \\ -x_2 & -x_3 & -x_1 & 0 & 0 & 0 \\ -x_3 & -x_1 & -x_2 & 0 & 0 & 0 \end{pmatrix}.$$

In coordinates, the non-vanishing Poisson brackets are

$$\begin{aligned} \{x_1, x_4\} &= \{x_2, x_6\} = \{x_3, x_5\} = x_1, \\ \{x_1, x_5\} &= \{x_2, x_4\} = \{x_3, x_6\} = x_2, \\ \{x_1, x_6\} &= \{x_2, x_5\} = \{x_3, x_4\} = x_3. \end{aligned}$$

General solution of (20) with $n = 3$ is

$$A = \begin{pmatrix} \alpha_0 & \alpha_2 & \alpha_1 & 0 & 0 & 0 \\ \alpha_1 & \alpha_0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_0 & \alpha_2 & \alpha_1 \\ 0 & 0 & 0 & \alpha_1 & \alpha_0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}, \quad (106)$$

and the corresponding set of functions in involution is generated by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (107)$$

The Hesse matrices \mathcal{H} of admissible Hamilton functions $H_0(x)$ satisfying (22) are of the form

$$\mathcal{H} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ h_2 & h_3 & h_1 & h_5 & h_6 & h_4 \\ h_3 & h_1 & h_2 & h_6 & h_4 & h_5 \\ h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\ h_5 & h_6 & h_4 & h_8 & h_9 & h_7 \\ h_6 & h_4 & h_5 & h_9 & h_7 & h_8 \end{pmatrix}. \quad (108)$$

The Hesse matrices of the commuting Hamilton functions $H_1(x)$, $H_2(x)$ are obtained by the simultaneous cyclic shifts of the triples $a = (h_1, h_2, h_3)$, $b = (h_4, h_5, h_6)$, and $c = (h_7, h_8, h_9)$.

The associated vector fields are produced by three linearly independent matrices

$$B_j = \alpha_j I + \beta_j A + \gamma_j A^2 = \mathcal{B}_j \oplus \mathcal{B}_j \quad (j = 0, 1, 2), \quad (109)$$

satisfying $B_j^2 = I$ and $B_0 + B_1 + B_2 = I$. Their diagonal 3×3 blocks are equal to:

$$\mathcal{B}_0 = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad (110)$$

$$\mathcal{B}_1 = \frac{1}{3} \begin{pmatrix} 1 & -2\omega & -2\omega^2 \\ -2\omega^2 & 1 & -2\omega \\ -2\omega & -2\omega^2 & 1 \end{pmatrix}, \quad \mathcal{B}_2 = \frac{1}{3} \begin{pmatrix} 1 & -2\omega^2 & -2\omega \\ -2\omega & 1 & -2\omega^2 \\ -2\omega^2 & -2\omega & 1 \end{pmatrix}, \quad (111)$$

where $\omega = \exp(2\pi i/3)$.

Lemma 8 in the present case of dimension $2n = 6$ gives the following results:

$$(J(x)\nabla^2 H)^2 = q_0(x)I + q_1(x)A + q_2(x)A^2 \quad (112)$$

$$= \begin{pmatrix} q_0(x) & q_1(x) & q_2(x) & 0 & 0 & 0 \\ q_2(x) & q_0(x) & q_1(x) & 0 & 0 & 0 \\ q_1(x) & q_2(x) & q_0(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & q_0(x) & q_1(x) & q_2(x) \\ 0 & 0 & 0 & q_2(x) & q_0(x) & q_1(x) \\ 0 & 0 & 0 & q_1(x) & q_2(x) & q_0(x) \end{pmatrix}. \quad (113)$$

The functions $q_0(x)$, $q_1(x)$, $q_2(x)$ satisfy, according to Lemma 9, following relations:

$$\nabla q_1(x) = A\nabla q_0(x), \quad \nabla q_2(x) = A^2\nabla q_0(x). \quad (114)$$

12. CONCLUSIONS

Completely integrable Hamiltonian systems lying at the basis of our constructions, seem to be worth studying on their own. In particular, their invariant n -dimensional varieties are intersections of n hyperquadrics in the $2n$ -dimensional space. It will be interesting to find out whether they are (affine parts) of Abelian varieties, that is, whether our systems are algebraically completely integrable. Still more interesting and intriguing are the algebraic-geometric aspects of the commuting systems of integrable maps introduced here. This will be the subject of our future research.

13. ACKNOWLEDGEMENTS

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